HADRONIC JOURNAL 48, 391 - 425 (2025)
DOI 10.29083/HJ.48.04.2025/SC391

FOUNDATIONS: RELATIVISTIC QUANTUM THEORY I MATHEMATICAL THEORY

T.L. Gill[1] and G.A. de Parga[2]
[1]Department of EECS, Mathematics and Computational Physics Laboratory, Howard University,
Washington, DC 20059
tgill@howard.edu
[2]Departmento de Física, Escuela Superior de Física y Matemáticas,
Instituto Politécnico Nacional COFAA, México
gadpau@hotmail.com

Received October 6, 2025
Revised October 10, 2025

Abstract

This is the first of a series on the foundations of quantum electrodynamics (QED). We first prove that: Minkowski's postulate is incompatible with the first postulate for two particles; analytic analysis of Dirac's equation leads to a new particle antiparticle separation, which reveals its time nonlocal nature; this also shows that the Pauli approximation is invalid for s-states and cut-offs are required before QED. We show Dirac's equation is not equivalent to the square root equation but is related by a unitary map. We then design the correct Hilbert space $KS^2[\mathbb{R}^n]$, for Feynman's path integral formulation of quantum theory and show it is not equivalent to $L^2[\mathbb{R}^n]$, the one for the Schödinger formulation, but is related by a unitary map. We also prove the existence of a universal clock and Santilli's dual of the real numbers to justify Feynman's view of antiparticles as particles going backwards in proper time. A major outcome of this study is that mathematical equivalence is not the same as physical equivalence and, careful physical analysis should be made before acceptance of mathematical concepts into physical theory.

*Corresponding author

Copyright © 2025 by Hadronic Press, Inc., Palm Harbor, FL 34684, USA

Introduction and Problems

We are ending the first quarter of the 21^{st} century and find ourselves still facing difficult problems left over from the 19^{th} century. Of particular concern, we have yet to obtain a relativistic version of Newtonian mechanics that is compatible with Maxwell's theory, and we can only partially claim a one-particle relativistic quantum theory (Dirac's equation requires holes for completion). The replacement of particles by fields have provided excellent agreement with experimental results. However, the methods used have led to an additional set of problems, that have defied resolution for almost five generations. The purpose of this study is to provide the mathematical foundations for a final solution. A major conclusion is that purely mathematical assumptions must receive careful vetting before acceptance as physically relevant and correct. The second paper is devoted to the classical foundations, while the third is devoted to the quantum foundations.

0.1 Framework and Historical Background

There are five postulates that provide the foundations for all of physical science:

(1) A real physical world, exists and is independent of our existence.

(2) This world is as it appears to us in our consciousness.

(3) This world is known via our senses (including surrogate instruments).

All the physical sciences are rooted in these postulates. Our scientific forefathers no longer exist, and yet, the laws they left us still exist (in some form), so the first postulate is clear. Without the second we can never be sure that our senses provide truth, making the third meaningless. Thus, we consider these postulates obvious.

0.1.1 Experimental Physics

Distortions and illusions about the world can appear in the consciousness of one or more us and not appear to others (the collective). Thus, we need a collective physical filter that allows us to distinguish physical truth from individual or group illusions. We refer to this filter as a scientific experiment.

Definition 0.1. *A scientific experiment is a controlled and reproducible experience. The basic postulate of experimental physics is that:*

(4) Objective reproducible empirical information about the physical world can be obtained through experimental investigation.

Definition 0.2. *We say that a physical subject is known when all possible cause effect-relations (of physical interest) can be a priori predicted under well-defined control conditions.*

0.1.2 Theoretical Physics

The objective of theoretical physics is to design and construct mathematical representations of the physical world. These representations must describe the cause-effect relationships observed in experiment, and must be physically and mathematically consistent, while using a minimal number of variables and parameters. The basic postulate of theoretical physics is as follows.

(5) Mathematics is the perfect intellectual design tool for the analysis and for certification of the consistency of representations of physical reality.

0.1.3 Newton-Maxwell Problem

The Newton-Maxwell problem was faced directly by Abraham, Einstein, Lorentz, Poincaré, Ritz, and other major thinkers of the 1900's. Starting with a complete analysis of the microscopic behavior of electrons and ions, Lorentz was able to explain all macroscopic properties of optics and electrodynamics (see [26], [27]). He was the first to obtain the transformations between different observers, showing how their relative velocities affected their results in different inertial frames. In his investigations, Poincaré discovered an error in the derivation and, after correction, named them Lorentz transformations. In addition, Poincaré observed that if time is treated as an imaginary coordinate, these Lorentz transformations form a group. (He also derived the proper time, which was later claimed by Minkowski.)

Einstein's work on the photo-electric effect, provided a different perspective. As noted by Brown [1], he did not believe that Maxwell's theory would survive the existence of photons, but considered Lorentz transformations fundamental. Thus, he derived them from kinematical arguments.

Einstein [6, 7] observed that a constant c appears in Maxwell's equations for all inertial observers. At that time, experimental information about the speed of light was limited, being restricted to macroscopic and astronomical studies. Einstein was the first to realize that a formal postulate on the

velocity of light is necessary. He proposed that all physical theories should satisfy the (now well-known) postulates of special relativity.

(**1**) The physical laws of nature and the results of all experiments were independent of the inertial frame of the observer (in which the experiment was performed).

(**2**) The speed of light in an empty space is constant and independent of the source or receiver motion (in an inertial frame).

Like others, Einstein realized that the idea of absolute space was neither knowable nor necessary. The first postulate abandons this idea. Einstein's second postulate appears to have abandoned the idea of absolute time.

Unlike Minkowski, Poincaré's insight and understanding of the difference between physics and mathematics helped him to resist the temptation to use the "physically unjustified" mathematical observation that time can be made a forth geometric coordinate as a (necessary) tool for the representation of physical reality. Minkowski later made this leap with much philosophical fanfare, but lacking any physical justification. Minkowski was a number theorist with few accomplishments in physics. In Walter's work, we find a clear discussion of Minkowski's physics background and his knowledge of Poincaré's work (see [46]).

Thus, we make explicit Minkowski's unacknowledged third postulate for Einstein's special theory of relativity.

(**3**) The correct implementation of the first two postulates requires that time be treated as a fourth coordinate, and the relationship between the components constrained to satisfy the natural invariance induced by the Lorentz group of electrodynamics (Minkowski space).

The physics community voted in favor of the Lorentz group as the proper transformation theory. Today, many believe that Newtonian theory is covered by Minkowski's version of the special theory. As will be shown later, the failure to conduct a complete (physical) analysis of the third postulate is at the root of all the major problems in the classical foundations.

0.2 Summary

In the first section we show that there are three representations of the proper time that follow from Einstein's version of the special theory. The first representation is well-known and is used to prove that Minkowski's postulate

is incompatible with Einstein's first postulate for two or more particles. We then discuss the center-of mass-problem due to Pryce [37], the many particle quantum problem discussed by Bakamjian and Thomas [2] and the no-interaction theorem first proven by Currie, Jordan, and Sudarshan [4]. We show that each of these foundational problems are a direct consequence of Minkowski's postulate. Finally, we prove that the second postulate is compatible with particle velocities larger than c, and that the assumption that the postulate holds for distant measurements is not valid.

In the second section, we briefly show that without the Minkowski postulate a new Klein-Gordon equation defined for all spins becomes possible. We then introduce an analytical separation method which shows that the Dirac equation can be separated directly revealing its time nonlocal properties. This approach allows us to show that the Pauli approximation cannot be used to study s-states in Hydrogen, and that cut-offs are required before QED. We conclude the section with a proof that the Dirac and square root equations although (mathematically) unitary equivalent, are not physically equivalent.

In the third section, we introduce the Henstock-Kurzweil integral (HK). It is very close to the one learned in calculus but extends the Lebesgue integral to include non-absolutely integrable functions (such as $e^{ix^2/2}$). We then design the natural Hilbert space for HK integrals, which allows us to construct the Feynman path integral in the manner he intended, while retaining both the physical intuition and computational advantage of his approach. This space $KS^2[\mathbb{R}^n]$ contains $L^2[\mathbb{R}^n]$ and the Schwartz distributions or generalized functions as continuous dense embeddings. From a physical point of view, the use of the HK-integral and $KS^2[\mathbb{R}^n]$ eliminates the extra time and effort required to first learn Lebesgue measure theory and then the theory of distributions.

In section four we use the third representation of proper time to prove the existence of a universal clock, under conditions weaker thaan the cosmological principle, which we call historical time (see [11] and [21]). We use this clock to construct Feynman's time-ordered operator calculus and use his theory to prove the last two remaining conjectures of Dyson concerning the foundations of QED.

1 Proper time and Minkowski's Postulate

Let m be the mass of a particle or the effective mass of a system of particles. We assume that the particle or system is defined in the phase space with variables $(t, \mathbf{x}, \mathbf{p})$ as seen by observer O, and variables $(t', \mathbf{x}', \mathbf{p}')$ as seen by observer O' (both inertial frames), where $\mathbf{x}, \mathbf{x}'$ is the position of the particle (or center of mass) and $(\mathbf{p}, \mathbf{p}')$ is the particle momentum (or center of mass momentum). If $\gamma^{-1}(\mathbf{v}) = \sqrt{1 - \frac{\mathbf{v}^2}{c^2}}$, $\mathbf{v} = \frac{d\mathbf{x}}{dt}$, where $\mathbf{v}$ $(\mathbf{v}')$ is the velocity of the particle, the first representation of the proper time is defined by our observers as.

$$d\tau = \gamma^{-1}(\mathbf{v})\, dt, \quad d\tau = \gamma^{-1}(\mathbf{v}')\, dt' \quad \underline{\text{proper time 1.}} \tag{1}$$

For the second representation, first square (1) to get:

$$d\tau^2 = dt^2 - \tfrac{1}{c^2} d\mathbf{x}^2, \quad d\tau^2 = dt'^2 - \tfrac{1}{c^2} d\mathbf{x}'^2.$$

Reaarrange terms to obtain:

$$d\tau^2 + \tfrac{1}{c^2} d\mathbf{x}^2 = dt^2, \quad d\tau^2 + \tfrac{1}{c^2} d\mathbf{x}'^2 = dt'^2 \Rightarrow$$
$$\sqrt{1 + \tfrac{\mathbf{u}^2}{c^2}}\, d\tau = dt, \ \mathbf{u} = \frac{d\mathbf{x}}{d\tau}, \quad \sqrt{1 + \tfrac{\mathbf{u}'^2}{c^2}}\, d\tau = dt', \ \mathbf{u}' = \frac{d\mathbf{x}'}{d\tau}.$$

If we let $b = \sqrt{c^2 + \mathbf{u}^2}$, $(b' = \sqrt{c^2 + \mathbf{u}'^2})$, then we have our second representation

$$cdt = bd\tau \quad \text{and} \quad cdt' = b'd\tau \quad \underline{\text{proper time 2.}} \tag{2}$$

For the third representation, use the relations $H = \gamma(\mathbf{v})\, mc^2$ for observer O and $H' = \gamma(\mathbf{v}')\, mc^2$ for observer O', to obtain:

$$d\tau = \tfrac{mc^2}{H} dt, \quad d\tau = \tfrac{mc^2}{H'} dt' \quad \underline{\text{proper time 3.}} \tag{3}$$

1.1 The Minkowski Problem

Minkowski used the first representation of proper time as a metric for his four-spacetime theory. As reported by Sommerfeld, Minkowski knew that the differential of proper time is not an exact one-form, as the value of τ depends on the forces acting on the particle (see the notes in [32]). Thus,

he introduced the co-moving observer as a substitute in order to use it as a metric. Einstein was the only one to openly object to Minkowski's postulate. He and Laub later published two papers on electrodynamics, which offered a different approach that was physically simpler, while not depending on the spacetime formalism (see [9], [10]). They argued that the spacetime formalism was complicated, required additional assumptions and did not add any new physics.

It is clear that Minkowski's postulate is mathematically very nice but it should have alarmed all physics thinkers, as "a co-moving observer cannot actually sit on the tangent plane of a bullet".

However, the idea of a four dimensional space-time was very attractive to mathematicians who had entered and dominated the field for over a decade. This mathematical authority had the effect of embedding Minkowski's postulate as natural part of Einstein's special theory. Consequently, intense physical investigation and justification normally given for new postulates did not occur. By the time problems in attempts to develop many particle relativistic quantum theories appeared, Minkowski's postulate was already sacred. For the general theory, Einstein only considered an approach that extended Minkowski's postulate (see Pais [31]). Unimpeachable physical problems only begin to appear after World War II, when new efforts to develop a many-particle theory were initiated.

1.1.1 The n-particle position Problem

In 1948 Pryce conducted the first study on the relativistic center-of-mass for two or more particles (see [37]). In order to reveal the first problem, we construct the many-particle clock for our O frame observer, we assume that n interacting particles have Hamiltonians H_i and a total Hamiltonian $H = \sum_{i=1}^{n} H_i$. The effective mass energy Mc^2, and total momentum $\mathbf{P}$, are defined as follows:

$$Mc^2 = \sqrt{H^2 - c^2\mathbf{P}^2}, \quad \mathbf{P} = \sum_{i=1}^{n} \mathbf{p}_i.$$

We can now also represent the Hamiltonian as $H = \sqrt{c^2\mathbf{P}^2 + M^2c^4}$. Pryce found that there three possible candidates for the center of mass. However, only one was canonical and available to all observers. The canonical center

of mass position $\mathbf{X}$ is defined by:

$$\mathbf{X} = \frac{1}{H}\sum_{i=1}^{n} H_i \mathbf{x}_i + \frac{c^2\,(\mathbf{S}\times\mathbf{P})}{H\,(Mc^2+H)}, \tag{4}$$

where $\mathbf{S}$ is the global spin of the system of particles relative to O (also see Longhi, Lusanna, and Pons [29]). Pryce then identified the first major problem, noting that the canonical representation of the center-of-mass cannot be the three-vector part of a four-vector. However, $\mathbf{X}$ is the canonical conjugate of $\mathbf{P}$, that is precisely the requirement for many-particle relativistic quantum theory. (It is clear that this conflicts with the third postulate.)

Five years after Pryce, Bakamjian and Thomas constructed a many-particle relativistic quantum theory using the canonical center of mass X. They explicitly showed that their theory satisfied Einstein's two postulates (see [2]). They conjectured that any attempt to enforce the third postulate would only be compatible with free particles (BT-conjecture).

In 1965 Currie, Jordan and Sudarshan [4], gave a proof of the BT-conjecture for two particles, which they renamed "The No-Interaction Theorem." It has since been extended to an arbitrary number of particles by Leutwyler [28].

Theorem 1.1. (No-Interaction Theorem) Suppose we have a system of n particles with phase space variables $\{(\mathbf{x}_i, \mathbf{p}_i)\}_{i=1}^{n}$ defined on $\mathbb{R}^{3n}\times\mathbb{R}^{3n}$ with the following properties:

1. The system provides a Hamiltonian representation.

2. The system has a canonical representation of the Poincaré group.

3. Each $\mathbf{x}_i$ is the vector of the four-vector.

Then these assumptions are only compatible with free particles.

Thus, we see that Minkowski's postulate appears to be the culprit that has hindered progress in both classical and quantum mechanics for over 120 years. The first hint that this may be the problem was actually given by Feynman, who showed that by allowing time to recover its role in physics as it appears in our consciousness, he could construct his operator calculus, his formulation of quantum mechanics, and QED. Feynman was criticized, but showed that this approach allowed him to give a physically intuitive and

a computationally efficient way of approaching QED. Although he shared the Nobel prize with Schwinger and Tomongona, for their work on the same problems, his approach is now taught in programs all over the world. The following theorem shows the importance of an early investigation of Minkowski's postulate.

Theorem 1.2. (Minkowski Incompatible Theorem) *The addition of Minkowski's postulate to the postulates of Einstein is incompatible for two or more particles,*

Proof. Let O and O' be two inertial observers. Without loss, we can assume that both clocks begin when their origins coincide and O' is moving with uniform velocity $\mathbf{v}$ as seen by O. Let two particles, each the source of an electromagnetic field, move with velocities $\mathbf{w}_i\,(i=1,2)$, as seen by O, and $\mathbf{w}'_i\,(i=1,2)$, as seen by O', such that:

$$\begin{aligned}\mathbf{x}'_i &= \mathbf{x}_i - \gamma(\mathbf{v})\,\mathbf{v}t + [\gamma(\mathbf{v})-1]\left(\mathbf{x}_i\cdot\mathbf{v}/\|\mathbf{v}\|^2\right)\mathbf{v},\\ \mathbf{x}_i &= \mathbf{x}'_i + \gamma(\mathbf{v})\,\mathbf{v}t' + [\gamma(\mathbf{v})-1]\left(\mathbf{x}'_i\cdot\mathbf{v}/\|\mathbf{v}\|^2\right)\mathbf{v} \quad \text{and,}\end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{w}'_i &= \mathbf{w}_i - \gamma(\mathbf{v})\,\mathbf{v} + [\gamma(\mathbf{v})-1]\left(\mathbf{w}_i\cdot\mathbf{v}/\|\mathbf{v}\|^2\right)\mathbf{v},\\ \mathbf{w}_i &= \mathbf{w}'_i + \gamma(\mathbf{v})\,\mathbf{v} + [\gamma(\mathbf{v})-1]\left(\mathbf{w}'_i\cdot\mathbf{v}/\|\mathbf{v}\|^2\right)\mathbf{v}.\end{aligned} \tag{6}$$

Thus, there is no problem in requiring the positions and velocities to be transformed as expected. However, when we try to transform the clocks, we see the problem at once because

$$t' = \gamma(\mathbf{v})\left(t - \mathbf{x}_1\cdot\mathbf{v}/c^2\right) \quad \text{and} \quad t' = \gamma(\mathbf{v})\left(t - \mathbf{x}_2\cdot\mathbf{v}/c^2\right). \tag{7}$$

This is impossible unless $\mathbf{x}_1\cdot\mathbf{v} = \mathbf{x}_2\cdot\mathbf{v}$ (all times). This means that the two observers cannot use their clocks to share information (with other observers) regarding two or more particles. It follows that if we attempt to replace $\mathbf{x}_i$ and $\mathbf{x}'_i$ with four vectors using t and t', then the first postulate fails. Thus, these three postulates are mathematically and physically incompatible. □

Remark 1.3. *The Lorentz transformations (equations (1.4) (1.5) and (1.6)) contain information about each observer and each particle, so it is not clear that they can be converted into the tensor notation that came into fashion after Sommerfeld simplified the (truly) complicated notation of Minkowski.*

1.2 Proper Time II and Einstein Compatibility Theorem

In this section, we first demonstrate how the second definition of proper time avoids the problems created by Minkowski's postulate.

1.2.1 One-Particle Clock

Assume a system of n-particles are observed by O, who is able to identify each particle and attach a vector $\mathbf{x}_i$ to the i^{th} particle, denoting its spacial distance to the origin. If $\mathbf{w}_i$ is the velocity of particle i as indicated by O, let $\gamma^{-1}(\mathbf{w}_i) = \sqrt{1 - \mathbf{w}_i^2/c^2}$. The i^{th} particle proper time is defined as

$$d\tau_i = \gamma^{-1}(\mathbf{w}_i)dt, \quad \mathbf{w}_i = \frac{d\mathbf{x}_i}{dt}, \quad d\tau_i^2 = dt^2 - \tfrac{1}{c^2}d\mathbf{x}_i^2. \tag{8}$$

Rewrite the last term to get:

$$dt^2 = d\tau_i^2 + \tfrac{1}{c^2}d\mathbf{x}_i^2, \Rightarrow cdt = \left(\sqrt{\mathbf{u}_i^2 + c^2}\right) d\tau_i, \quad \mathbf{u}_i = \frac{d\mathbf{x}_i}{d\tau_i} = \gamma(\mathbf{w}_i)\mathbf{w}_i. \tag{9}$$

If we let $b_i = \sqrt{\mathbf{u}_i^2 + c^2}$, the second term in equation (9) becomes $cdt = b_i d\tau_i$. This leads to the following new identity:

$$\frac{1}{c}\frac{d}{dt} \equiv \frac{1}{b_i}\frac{d}{d\tau_i}, i = 1, \ldots, n. \tag{10}$$

Thus, we obtained an interesting set of relationships between the proper time of each particle and that of the observer. Applying this to $\mathbf{x}_i$, we obtain another set of relationships showing that the tangent space is left invariant in all cases.

$$\frac{\mathbf{w}_i}{c} = \frac{1}{c}\frac{d\mathbf{x}_i}{dt} \equiv \frac{1}{b_i}\frac{d\mathbf{x}_i}{d\tau_i} = \frac{\mathbf{u}_i}{b_i}, i = 1, \ldots, n. \tag{11}$$

The new particle coordinates are $(\mathbf{x}_i, \tau_i)$. In this representation, $\mathbf{x}_i$ is uniquely defined relative to O, whereas τ_i is uniquely defined for each particle.

Remark 1.4. *Equation (11) shows that the second postulate is compatible with speeds higher then the speed of light as seem by inertial observers. It also means that the use of $\beta = \mathbf{w}_i/c$ as a parameter for distant measurements is no-longer reliable and that time dilation is a physically incorrect concept.*

As the momentum is $\mathbf{p}_i = m_i\gamma(\mathbf{w}_i)\mathbf{w}_i$ and $\mathbf{u}_i = \gamma(\mathbf{w}_i)\mathbf{w}_i$, we can also represent the momentum as $\mathbf{p}_i = m_i\mathbf{u}_i$. It follows that the particle phase space is also left invariant. Because m_i is the rest mass, we see that the concept of relativistic mass increase is also a physically incorrect concept.

Theorem 1.5. Einstein Compatibility Theorem *If the observer time is used at the global level and each particle's proper time is used at the local level, then Einstein's postulates hold and the transformations preserving the first postulate are:*

$$b_i{}' = \gamma(\mathbf{v})\left(b_i - \mathbf{u}_i \cdot \mathbf{v}/c\right)$$

and

$$b_i = \gamma(\mathbf{v})\left(b_i' + \mathbf{u}'_i \cdot \mathbf{v}/c\right).$$

Proof. If we use each particle proper time then, for observer O'

$$\frac{\mathbf{w}_i'}{c} = \frac{1}{c}\frac{d\mathbf{x}_i'}{dt} \equiv \frac{1}{b_i'}\frac{d\mathbf{x}_i'}{d\tau_i} = \frac{\mathbf{u}_i'}{b_i'}, i = 1, \ldots, n.$$

From $t' = \gamma(\mathbf{v})\left(t - \mathbf{x}_i \cdot \mathbf{v}/c^2\right)$ we have $cdt' = \gamma(\mathbf{v})\left(cdt - d\mathbf{x}_i \cdot \mathbf{v}/c\right)$. Using the two identities $cdt' = b_i' d\tau_i$ and $cdt = b_i d\tau_i$, we obtain:

$$b_i{}' d\tau_i = \gamma(\mathbf{v})\left(b_i d\tau_i - d\mathbf{x}_i \cdot \mathbf{v}/c\right)$$

Dividing by $d\tau_i$, we see that the transformations between b_i and b_i' are

$$b_i{}' = \gamma(\mathbf{v})\left(b_i - \mathbf{u}_i \cdot \mathbf{v}/c\right).$$

By a similar calculation, we obtain the reverse relationship:

$$b_i = \gamma(\mathbf{v})\left(b_i' + \mathbf{u}'_i \cdot \mathbf{v}/c\right).$$

It now follows that the two postulates are valid for all observers. □

Each observer can now use their clock and the canonical center of mass to quantize the n-particle system (following [2]).

Remark 1.6. *We note that the second postulate is still valid for all observers, and is necessary for observers to share information, insuring the first postulate.*

2 Quantum Problems

This section is devoted to the analysis of a number of missed and/or misunderstood mathematical issues in the foundations of relativistic quantum theory. We begin with a new look at the Dirac equation and investigate what happens when the potential is viewed as an addition to the mass (as often occurs classically). This leads us to a new Klein-Gordon equation, without changing the wave function and is compatible with all particle spin values. We then conduct an analytical analysis of the Dirac equation and show that it can be diagonalized exactly without changing the wave function (in distinction to the Foldy-Wouthuysen method [13]). It allows us to directly demonstrate that the Pauli approximation cannot be used to study the s-states in hydrogen. We finally show that although related by a unitary transformation, the Dirac and square root operators are not physically equivalent. Thus, mathematical equivalence may have nothing to do with the physics.

2.1 The Dirac Equation

After Palui proposed his exclusion principle in 1925 ([35]), Dirac was able to find the first physically viable method around the square-root problem in 1926 (see [5]). Raised in the British tradition of quaternions, Dirac knew that quaternions could be used to generate a spin-1/2 representation for the rotation group. He used Pauli matrices to write the square-root operator as a perfect square:

$$\sqrt{c^2\mathbf{p}^2 + m^2c^4} = \sqrt{\left[c\boldsymbol{\alpha}\cdot\mathbf{p} + \boldsymbol{\beta}mc^2\right]^2}, \quad \text{where,} \quad \boldsymbol{\beta} = \begin{bmatrix} I_2 & 0 \\ 0 & -I_2. \end{bmatrix} \tag{12}$$

and the matrix $\boldsymbol{\alpha}$ is defined by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, with

$$\alpha_i = \begin{pmatrix} \mathbf{0} & \sigma_i \\ \sigma_i & \mathbf{0} \end{pmatrix}, \ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

thus obtaining an alternative representation of equation (12), now known as the Dirac equation:

$$i\hbar\frac{\partial\Psi}{\partial t} = \left[c\boldsymbol{\alpha}\cdot\mathbf{p} + \boldsymbol{\beta}mc^2\right]\Psi \tag{13}$$

We must now view Ψ as a vector-valued function or spinor. In other words, $\Phi \in L^2(\mathbb{R}^3, C^4) = L^2(\mathbb{R}^3) \otimes C^4$ is a four-component column vector $\Psi = (\psi_1, \psi_2, \phi_1, \phi_2)^t$. In this representation, $\psi = (\psi_1, \psi_2)^t$ is the particle component and $\phi = (\phi_1, \phi_2)^t$ is the antiparticle component. Dirac's equation provided the correct spin value for the electron and the known (at that time) spectrum for hydrogen. His equation also predicted negative energy solutions, which led Dirac to propose the hole theory of vacuum. In this view, a hole is an electron with a negative energy and positive charge. This means that Dirac's theory cannot be applied to a single particle. However, the later discovery of the positron confirmed this approach. Schweber [45] provides a good discussion of the difficulties that eventually lead to QED.

2.1.1 The Dirac-Klein-Gordon Connection

When the mass is constant it was shown by Case [3] that a Foldy-Wouthuysen (unitary) transformation can be constructed for particles of spin 0 and 1, Then Pursey [34] showed that such a transformation existed for all values of spin. This means that we can obtain a Dirac equation for all spin values. In this section, we show that, unconstrained by the Minkowski postulate, there is a new and important relationship between the Dirac and Klein-Gordon equation. Let $E[\Psi] = [c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \boldsymbol{\beta} mc^2 + V]\,\Psi$, where $V = e\phi$, $\partial V/\partial t = 0$, is the potential energy and $\boldsymbol{\pi} = \mathbf{p} - e/c\mathbf{A}$. Now note that $\boldsymbol{\beta} mc^2 + V = \boldsymbol{\beta}(mc^2 + \boldsymbol{\beta} V)$, such that

$$\left[E - c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} - \boldsymbol{\beta}(mc^2 + \boldsymbol{\beta} V)\right] \Psi = \mathbf{0} \tag{14}$$

Now multiply on the left by $[E + c\boldsymbol{\alpha} \cdot \boldsymbol{\pi} + \boldsymbol{\beta}(mc^2 + \boldsymbol{\beta} V)]$, doing the standard computations:

$$e\boldsymbol{\alpha} \cdot [E\mathbf{A} - \mathbf{A}E] = ie\hbar\boldsymbol{\alpha} \cdot \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \quad \text{and}$$

$$c\boldsymbol{\beta}\boldsymbol{\alpha} \cdot [\boldsymbol{\beta} V\mathbf{p} - \mathbf{p}\boldsymbol{\beta} V] = ie\hbar c\boldsymbol{\alpha} \cdot \nabla\phi,$$

to obtain:

$$\begin{aligned} &\left\{E^2 - c^2\boldsymbol{\pi}^2 + e\hbar c\boldsymbol{\Sigma} \cdot \mathbf{B} + ie\hbar c\boldsymbol{\alpha} \cdot \mathbf{E}\right. \\ &\left. + 2i\hbar c\boldsymbol{\alpha} \cdot \nabla V - 2cV\boldsymbol{\alpha} \cdot \mathbf{p} - (mc^2 + \boldsymbol{\beta} V)^2\right\} \Psi = \mathbf{0}. \end{aligned} \tag{15}$$

Where $\mathbf{E} = -\partial\mathbf{A}/\partial ct - \nabla\phi$ is the electric field. We thus obtain a Klein-Gordon equation valid for all values of spin:

$$\begin{aligned} -\hbar^2\frac{\partial^2\Psi}{\partial t^2} = \{c^2\boldsymbol{\pi}^2 - e\hbar c\boldsymbol{\Sigma}\cdot\mathbf{B} - ie\hbar c\boldsymbol{\alpha}\cdot\mathbf{E} \\ -2i\hbar c\boldsymbol{\alpha}\cdot\nabla V - 2cV\boldsymbol{\alpha}\cdot\mathbf{p} + (mc^2+\boldsymbol{\beta}V)^2\}\,\Psi. \end{aligned} \tag{16}$$

When $\mathbf{A}$ and V are zero, the above becomes the standard KG-equation. Belief in the correctness of the Minkowski postulate dictated the current manner of introducing minimal coupling. This approach is discussed in Schiff ([40], see pg. 320 eq. 42.9 and pg. 329 eq. 43.25). Schweber provides a good discussion of the interpretation problems because of the possible negative values of the natural probability density (see [44]).

2.1.2 Analytical Analysis of the Dirac Equation

A complete understanding of the Dirac equation was only possible after the development of the geometric algebra. We now see that Dirac used Lorentz-invariant Clifford algebra to represent the complex algebra of observables for the electron (see Hestenes [20]). However, it is apparently believed that when interaction is present, a complete separation of the particle and antiparticle components of his equation without approximation is not possible. The algebraic studies of Foldy-Wouthuysen [13], Feynman and Gell-Mann [12], and Pauli [36] along with the various approximation approaches in the literature, may have supported this idea.

The first exact analytical separation of Dirac's equation, with minimal coupling, was reported by Gill et. al. (2005). Because this is important for our physical interpretation of Dirac's equation, the correct physical interpretation of the zitterbewegung, the correct physical relation between Dirac's equation and the square root equation and to understand the physical failure of the Palui approximation for the study of s-states in hydrogen, we provide a brief outline of this method (see [19] for details).

If $\mathbf{A}(\mathbf{x},t)$ and $V(\mathbf{x})$ are vector and scalar potentials, respectively with standard identification $\mathbf{p} \rightarrow \boldsymbol{\pi} = \mathbf{p} - (e/c)\mathbf{A}$ and write (13) in a two-component form:

$$i\hbar\frac{\partial\boldsymbol{\psi}}{\partial t} - \left(mc^2+V\right)\boldsymbol{\psi} = c\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\boldsymbol{\varphi}, \tag{17}$$

and

$$i\hbar\frac{\partial\boldsymbol{\varphi}}{\partial t}+\left(mc^2-V\right)\boldsymbol{\varphi}=c\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\boldsymbol{\psi} \tag{18}$$

where the vectors $\boldsymbol{\psi}=[\psi_1,\psi_2]^t$ and $\boldsymbol{\varphi}=[\varphi_1,\varphi_2]^t$ represent the particle and antiparticle components $\Psi=[\boldsymbol{\psi},\boldsymbol{\varphi}]^t$. These two equations can be separated using the method of integrating factors (as in elementary differential equations). We solve for $\boldsymbol{\varphi}$ in the second equation and use the result in the first to obtain an equation that depends only on $\boldsymbol{\psi}$. Thus, we need to determine δ such that.

$$e^{-\delta t}i\hbar\frac{\partial}{\partial t}\left[\boldsymbol{\varphi}e^{\delta t}\right]=i\hbar\frac{\partial\boldsymbol{\varphi}}{\partial t}+\left(mc^2-V\right)\boldsymbol{\varphi}.$$

The left-hand side was simplified and compared to obtain that $\delta=\frac{1}{i\hbar}\left(mc^2-V\right)$. We now assume that $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ vanish at $-\infty$ and integrate to obtain.

$$\begin{aligned}&i\hbar\frac{\partial\boldsymbol{\psi}}{\partial t}-\left(mc^2+V\right)\boldsymbol{\psi}=\\&c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left\{\int_{-\infty}^{t}\exp\left\{\tfrac{i}{\hbar}\left(mc^2-V\right)\left(t-\tau\right)\right\}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\boldsymbol{\psi}(\mathbf{x},\tau)d\tau\right\}.\end{aligned} \tag{19}$$

Using the same method, we obtain our second equation for $\boldsymbol{\varphi}$:

$$\begin{aligned}&i\hbar\frac{\partial\boldsymbol{\varphi}}{\partial t}+\left(mc^2-V\right)\boldsymbol{\varphi}=\\&c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left\{\int_{-\infty}^{t}\exp\left\{-\tfrac{i}{\hbar}\left(mc^2+V\right)\left(t-\tau\right)\right\}\left(\boldsymbol{\sigma}\cdot\pi\right)\boldsymbol{\varphi}(\mathbf{x},\tau)d\tau\right\}.\end{aligned} \tag{20}$$

Equations (19) and (20) now decomposed $L^2(\mathbb{R}^3,C^4)$ into a direct sum $L^2(\mathbb{R}^3,C^4)=L^2(\mathbb{R}^3,C^2)\oplus L^2(\mathbb{R}^3,C^2)$. For later use, the probability density for $\boldsymbol{\psi}$ is:

$$\rho_{\boldsymbol{\psi}}(t)=\left|\boldsymbol{\psi}\left(\mathbf{x},t\right)\right|^2+\left|c\int_{-\infty}^{t}\exp\left\{\tfrac{i}{\hbar}\left(mc^2-V\right)\left(t-\tau\right)\right\}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\boldsymbol{\psi}\left(\mathbf{x},\tau\right)d\tau\right|^2. \tag{21}$$

Thus, we obtain a complete analytic diagonalization of the particle and antiparticle wave functions (see [19]). The decomposition is analytic compared to Foldy and Wouthuysen's group theoretical framework in that the wave function is not transformed (see [13]). We discuss this in further detail, after studying the square root operator.

2.1.3 Pauli Approximation

For the eigenvalue problem, equations (17) and (18) can be written as.

$$\left(E - V - mc^2\right)\psi = c\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\varphi,$$
$$\left(E - V + mc^2\right)\varphi = c\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\psi.$$

Using

$$\varphi = c\left(E - V + mc^2\right)^{-1}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\psi,$$

in the first equation and

$$\psi = c\left(E - V - mc^2\right)^{-1}\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\varphi,$$

in the second equation. After simplification, we obtain:

$$\left(E - V - mc^2\right)\psi = \frac{c^2\left(\boldsymbol{\sigma}\cdot\mathbf{p}V\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(E - V + mc^2\right)^2}\psi + \frac{c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(E - V + mc^2\right)}\psi \qquad (22)$$

and

$$\left(E - V + mc^2\right)\varphi = \frac{c^2\left(\boldsymbol{\sigma}\cdot\mathbf{p}V\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(E - V - mc^2\right)^2}\varphi + \frac{c^2\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{\left(E - V - mc^2\right)}\varphi. \qquad (23)$$

Equations (22) and (23) first appeared in Williams in 1940 (see [47]). He was a student of Slater, who included them in the appendix of his 1960 book (see [41]). We call these Williams-Slater equations. As will be seen, they are important for understanding the mathematical foundations of QED. In equation (22), $\left(E - V + mc^2\right)^{-2}$ is replaced by 0 and $E - V + mc^2$ is approximated by $2mc^2$ to obtain the Pauli's equation:

$$\left(E - V - mc^2\right)\psi = \frac{\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{2m}\psi.$$

The probability density for this equation now becomes:

$$\rho_\psi = \left|\psi\left(\mathbf{x}\right)\right|^2 + \left|\frac{\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)}{2mc}\psi\left(\mathbf{x}\right)\right|^2$$

If we use

$$\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right)\left(\boldsymbol{\sigma}\cdot\boldsymbol{\pi}\right) = \pi^2 - \frac{e\hbar}{c}\boldsymbol{\sigma}\cdot\mathbf{B},$$

We obtain the standard form:

$$E\psi = \left[\frac{\boldsymbol{\pi}^2}{2m} - \frac{e\hbar(\boldsymbol{\sigma}\cdot\mathbf{B})}{2mc} + V + mc^2\right]\psi \tag{24}$$

In the case of interest, $V_0 = -mc^2 r_0/r$, $\mathbf{A} = \boldsymbol{\mu}_p \times \mathbf{r}/r^3$, with the spin oriented along the z-axis; thus, $A_r = A_\theta = 0$ and $A_\phi = \frac{2\mu_p s_p \sin\theta}{r^2}$. Therefore, we see that $\boldsymbol{\pi}^2 = \mathbf{p}^2 + \mathbf{A}^2$. Recall that QED provides corrections to the Dirac equation for s-states in hydrogen based on the Pauli approximation. Thus, it is important to know the extent to which we can draw any conclusions from the Pauli equation regarding the behavior of the Dirac equation in s-states. Recalling that a Dirac electron has a finite probability of being at the origin in the s-state, we see that the following invalidates Pauli's approximation.

1. $(E - V + mc^2)$ cannot be approximated by $2mc^2$ for s-states.
2. Because $\mathbf{A}$ has an r^{-2} term and $\mathbf{A}^2$ has an r^{-4} term they cannot be omitted for s-state calculations.
3. The use of $\rho_\psi = |\psi(\mathbf{x})|^2$ because this as the density is incorrect and the corrected density $\rho_\psi = |\psi(\mathbf{x})|^2 + \left|\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\pi})}{2mc}\psi(\mathbf{x})\right|^2$ cannot be used because of (2).

To resolve these difficulties, we note that the binding energy in hydrogen is 13eV, while the rest mass of the electron is 5×10^5eV, so the ratio is 2.6×10^{-5}. Thus, we can replace $E - V + mc^2$ with $2mc^2 + \frac{e^2}{r}$ with little error, but a cut-off is required (see [19] for details). The important conclusion is that any analysis of the Dirac hydrogen atom problem for s-states requires a cut-off prior to any QED calculations.

Remark 2.1. *We note that, for the p-state each of these terms are small, so the Pauli approximation is valid for p-state calculations*

2.1.4 The Square Root Equation

In 2005, Gill and Zachary [18] used the theory of fractional powers of linear operators to construct an (analytic) representation of the square-root energy operator valid for all spin values. A better representation was obtained by

Samko (see [38] pg. 270). If Δ is the Laplacian and $\omega = mc/\hbar$, we have:

$$\begin{aligned} &\beta\hbar c\left[\sqrt{\omega^2 I - \Delta}\right]\psi(\mathbf{x},t) = \beta mc^2\psi(\mathbf{x},t) \\ &-\frac{\beta\hbar c}{2\pi^2}\left\{\int_{\mathbb{R}^3}\frac{K_2[\omega|\mathbf{x}-\mathbf{y}|]}{|\mathbf{x}-\mathbf{y}|^2}\left[\psi(\mathbf{x},t)-\psi(\mathbf{y},t)\right]d\mathbf{y}\right\}. \end{aligned} \tag{25}$$

The integral is over $\mathbb{R}^3$ showing in what sense the square root operator is nonlocal in space. For a better understanding of this extended object, we require more information about the Bessel functions $K_2[\omega|\mathbf{y}|]$, $K_1[\omega|\mathbf{y}|]$, $K_{1/2}[\omega|\mathbf{y}|]$ and $K_0[\omega|\mathbf{y}|]$ (see Gradshteyn and Ryzhik [15]).

$$K_2[\omega|\mathbf{y}|] = K_0[\omega|\mathbf{y}|] + \frac{K_1[\omega|\mathbf{y}|]}{[\omega|\mathbf{y}|]} \quad \text{and}$$
$$\frac{K_{1/2}[\omega|\mathbf{y}|]}{[\omega|\mathbf{y}|]^{1/2}} = \sqrt{\tfrac{\pi}{2}}\frac{exp\{-\omega|\mathbf{y}|\}}{[\omega|\mathbf{y}|]}.$$

Since the integral of $|\mathbf{x}-\mathbf{y}|^{-2}$ is finite, the effective singularity properties for $0 < z \ll 1$ are:

$$\left.\begin{array}{c} \frac{K_1[z]}{z} = C_1\left[1+\theta_1(z)\right]z^{-2} \\ \frac{K_{1/2}[z]}{z^{1/2}} = \sqrt{\frac{\pi}{2}}\left[1+\theta_2(z)\right]z^{-1} \\ K_0[z] = C_0\left[1+\theta_0(z)\right]\ln z^{-1} \end{array}\right\}.$$

We see that K_1 diverges at 0 like z^{-2}. We conjecture that this term is the particle representation for the strong interaction. K_0 is an an integrable singularity. (We include $K_{1/2}$ for comparison, as it is essentially the well-known Yakawa potential, expected to account for short-range nuclear interactions). The effective properties for $z \gg 1$ are:

$$\left.\begin{array}{c} \frac{K_1[z]}{z} = C_1\left[1+\Theta_1(z)\right]z^{-3/2}e^{-z} \\ \frac{K_{1/2}[z]}{z^{1/2}} = \sqrt{\frac{\pi}{2}}\left[1+\Theta_{1/2}(z)\right]z^{-1}e^{-z} \\ K_0[z] = C_0\left[1+\Theta_0(z)\right]z^{-1/2}e^{-z} \end{array}\right\}.$$

In this region, we have a cut-off range and the strength of the interaction reverses, with the K_0 having the longest tail. We conjecture that this term is the particle representation for the weak interaction. It is clear that the square root operator is nonlocal in space and can be treated as an identity operator outside a few Compton wavelengths.

2.1.5 Discussion I

The important conclusion from the previous two sections is that the Dirac and square root operators are not physically equivalent. The Dirac operator is nonlocal in time, whereas the square root operator is nonlocal in space. It is shown in [19] that the correct physical interpretation of the zitterbewegung and the fact that the expected instantaneous value of the velocity measurement of a Dirac particle is $\pm c$ is that it instantaneously jumps between $+c$ and $-c$ at each point in time to make it appear as a point in space.

We must now account for the fact that the Foldy-Wouthuysen is a unitary transformation mapping the coupled particle-antiparticle Dirac operator to the uncoupled particle-antiparticle square root operator. From equations (2.9) and (2.10), we see that the Foldy-Wouthuysen unitary operator maps a nonlocal-in-time operator to a nonlocal-in-space operator, so that the unitary property of an operator is not sufficient to give confidence in a physical interpretation. We should also remember that the concept of unitary equivalence is purely mathematical and taken as a tool for physics without complete physical justification.

Remark 2.2. *The Schödinger representation (on $L^2[\mathbb{R}^3]$) and the Heisenberg representation $\ell_2[\mathbb{R}^3]$ were known to be physically equivalent before they were proven to also be mathematical equivalent (i.e, related by a unitary mapping) by von Neumann. However, there are an infinite number of Hilbert spaces related to $\ell_2[\mathbb{R}^3]$ by a unitary mapping, which have nothing to due with quantum theory. In a physically justified sense, $L^2[\mathbb{R}^3]$ is not the correct Hilbert space for Feynman's path integral formulation of quantum mechanics. This is discussed in the next section. This space serves as another counter example to the notion that mathematical equivalence means physical equivalence.*

3 Hilbert Space for Feynman's Formulation of Quantum Mechanics

We begin this section with a brief introduction to the Henstock-Kurzweil integral (HK). This integral is well defined for operator-valued functions that may not be separably valued (where both the Bochner and Pettis integrals are undefined). Intuitively, one uses a version of the Riemann integral (of calculus) with the interior points chosen first, whereas the size of the base

rectangle around any interior point is determined by an arbitrary positive function defined at that point. This integral was discovered independently by Henstock [22] and Kurzweil [25]. Our aim is to provide a sense of the conceptual and technical simplicity of the HK integral. Those interested in more details and proofs are directed to Gill and Zachary [17]. Let $\mathcal{H}$ be a separable Hilbert space and let $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Let $[a,b] \subset \mathbb{R}$ and, for each $t \in [a,b]$, let $H(t) \in L(\mathcal{H})$ be a given family of operators.

Definition 3.1. *Let $\delta(t)$ map $[a,b] \to (0,\infty)$, and let $\mathbf{P} = \{t_0, \tau_1, t_1, \tau_2, \cdots, \tau_n, t_n\}$, where $a = t_0 \leqslant \tau_1 \leqslant t_1 \leqslant \cdots \leqslant \tau_n \leqslant t_n = b$. We call $\mathbf{P}$ a HK-δ partition for $[a,b]$ provided that for $0 \leqslant i \leqslant n-1$, $t_i, t_{i+1} \in (\tau_{i+1} - \delta(\tau_{i+1}), \tau_{i+1} + \delta(\tau_{i+1}))$.*

Definition 3.2. *The family $H(t)$, $t \in [a,b]$, is said to have a (uniform) HK integral if there is an operator $Q[a,b]$ in $L(\mathcal{H})$ such that, for each $\varepsilon > 0$, there exists a function δ from $[a,b] \to (0,\infty)$ such that, whenever $\mathbf{P}$ is a HK-δ partition for $[a,b]$, then*

$$\left\| \sum_{i=1}^{n} \Delta t_i H(\tau_i) - Q[a,b] \right\| < \varepsilon.$$

In this case, we write

$$Q[a,b] = (HK) \int_a^b H(t)dt.$$

Example 3.3. *The following example shows that there are HK-integrable functions, that are not Lebesgue integrable. If $F'(t) = 2tsin(1/t^2) - (2/t)cos(1/t^2)$, then for $t \in (0,1)$ and $F'(0) = 0$. It is easy to see that $F(t) = t^2 sin(1/t^2)$, so that*

$$(HK) \int_0^1 \left(2t \sin \frac{1}{t^2} - 2\frac{1}{t} \cos \frac{1}{t^2} \right) dt = \sin 1.$$

The following theorem explains the difference between the two integrals.

Theorem 3.4. *Let $f(t) : [a,b] \to \mathbb{R}$.*

1. *If $f(t)$ is L-integrable on $[a,b]$, then it is HK-integrable on $[a,b]$ and: HK-$\int_a^b f(t)dt$ = L-$\int_a^b f(t)dt$.*

2. *If $f(t)$ is HK-integrable on $[a,b]$, then $\sup_{t>a} \left| \int_a^t f(s)ds \right| < \infty$.*

3.1 The KS-Hilbert Space

In this section, we construct a particular (separable) Hilbert space $KS^2[\mathbb{R}^n]$. This space is designed to contain the class of HK-integrable functions, the space of the test functions $\mathcal{D}[\mathbb{R}^n]$, the space of finitely additive measures on $\mathbb{R}^n$, $\mathfrak{M}[\mathbb{R}^n]$ and each of the $L^p[\mathbb{R}^n]$ spaces, for $1 \leqslant p \leqslant \infty$. Furthermore, each of the latter class spaces will be contained in $KS^2[\mathbb{R}^n]$ as a continuous dense and compact embeddings (e.g., weakly convergent sequences in each of the above spaces are strongly convergent in $KS^2[\mathbb{R}^n]$). In particular, $KS^2[\mathbb{R}^n]$ is perfect for constructing Feynman's path integral formulation of quantum mechanics (in the form close to his suggestion, while maintaining the physical intuition and computational advantages).

3.2 The Inner Product

It is not well-known in the mathematical community, but there are two inner products that may be defined on the class of functions $\mathbb{C}_c^\infty[\mathbb{R}^n]$, $n \in \mathbb{N}$. The first is well-known, if μ is a measure on $\mathbb{R}^n$, we have:

$$(f,g)_1 = \int_{\mathbb{R}^n} f(\mathbf{x})g^*(\mathbf{x})d\mu(\mathbf{x}),$$

and

$$(f,g)_2 = \int_{\mathbb{R}^n \times \mathbb{R}^n} f(\mathbf{x})g^*(\mathbf{y})d\mu(\mathbf{x})d\mu(\mathbf{y}).$$

In the first case,

$$||f||_1^2 = \int_{\mathbb{R}^n} |f(\mathbf{x})|^2 d\mu(\mathbf{x}), \tag{26}$$

while in the second case,

$$||f||_2^2 = \left| \int_{\mathbb{R}^n} f(\mathbf{x})d\mu(\mathbf{x}) \right|^2, \tag{27}$$

It is clear that (26) is a norm for absolutely integrable functions, while (27) is a norm for non-absolutely integrable functions. This latter form of the inner product was first (indirectly) revealed in the work of Kuelbs [24].)

3.3 The Kuelbs-Steadman Hilbert space $KS^2[\mathbb{R}^n]$

We know from the work of Henstock [22] that, in the one-dimensional case we can norm the non-absolutely integrable functions on $[0, \infty)$ by:

$$||f|| = sup_{t>0} \left| \int_0^t f(s)ds \right|$$

and the closure in this norm contains some distributions but no other information is known. Thus, we imposed the following (physically motivated) design specifications on our Hilbert space:

1. The space should contain all HK-integrable functions.
2. The space should contain all functions in $L^2[\mathbb{R}^n]$ as a continuous dense embedding.
3. The space should contain the space of test functions as a continuous dense embedding.

To construct our space, fix n, and let $\mathbb{Q}^n$ be the set all vectors $\{\mathbf{x} = (x_1, x_2 \cdots, x_n) \in \mathbb{R}^n\}$ such that x_i is rational for each i. This is a countable dense set in $\mathbb{R}^n$, so we can arrange it as $\mathbb{Q}^n = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \cdots\}$. For each i and j, let $\mathbf{B}_i(\mathbf{x}_j)$ be a closed cube centered at $\mathbf{x}_j$ of side $r_i = 2^{-i}, i \in \mathbb{N}$ parallel to the coordinate axis. Now choose an order such that the set $\{\mathbf{B}_k(\mathbf{x}_k),\ k \in \mathbb{N}\}$ contains all closed cubes $\{\mathbf{B}_i(\mathbf{x}_j)\ \ |(i,j) \in \mathbb{N} \times \mathbb{N}\}$ centered at a point in $\mathbb{Q}^n$. Let $\mathcal{E}_k(\mathbf{x}) = 1$ if $\mathbf{x} \in \mathbf{B}_k(\mathbf{x}_k)$ and be zero otherwise, so that $\mathcal{E}_k(\mathbf{x})$ is in $L^p[\mathbb{R}^n]$ for $1 \leqslant p \leqslant \infty$. We define $F_k(\,\cdot\,)$ on $L^1[\mathbb{R}^n]$ by

$$F_k(f) = \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}. \tag{28}$$

It is clear that $F_k(\,\cdot\,)$ is a bounded linear functional on $L^p[\mathbb{R}^n]$ for each k, $\|F_k\|_\infty \leqslant 1$ and if $F_k(f) = 0$ for all k, $f = 0$ so that $\{F_k\}$ is fundamental on $L^p[\mathbb{R}^n]$ for $1 \leqslant p \leqslant \infty$. Let $t_k > 0$, $\sum_{k=1}^{\infty} t_k = 1$ and define a measure $d\mathbf{P}(\mathbf{x}, \mathbf{y})$ on $\mathbf{R}^n \times \mathbf{R}^n$ as follows:

$$d\mathbf{P}(\mathbf{x}, \mathbf{y}) = \left[\sum_{k=1}^{\infty} t_k \mathcal{E}_k(\mathbf{x}) \mathcal{E}_k(\mathbf{y})\right] d\mathbf{x} d\mathbf{y}.$$

We can now define an inner product $(\cdot)$ on $\mathbf{L}^1[\mathbf{R}^n]$ by

$$\begin{aligned}(f,g) &= \int_{\mathbb{R}^n\times\mathbb{R}^n} f(\mathbf{x})g(\mathbf{y})^* d\mathbf{P}(\mathbf{x},\mathbf{y}) \\ &= \sum_{k=1}^{\infty} t_k \left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right]\left[\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{y}) g(\mathbf{y}) d\mathbf{y}\right]^*.\end{aligned} \tag{29}$$

We call the completion of $L^1[\mathbb{R}^n]$, with the above inner product, the Kuelbs-Steadman space $KS^2[\mathbb{R}^n]$. To see that it contains HK-integrable functions, let f be HK-integrable on $\mathbb{R}^n$ then:

$$\|f\|_{KS}^2 = \sum_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^2 \leqslant \sup_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^2 \leqslant \left|\int_{\mathbb{R}^n} f(\mathbf{x}) d\mathbf{x}\right|^2 < \infty,$$

so $f \in KS^2[\mathbb{R}^n]$.

Definition 3.5. *We say that $\mathcal{B} \subset \mathcal{H}$ is a continuous dense embedding if:*

1. *$\mathcal{B}$ is dense in $\mathcal{H}$.*
2. *For each $f \in \mathcal{B}$, $||f||_{\mathcal{H}} \leq ||f||_{\mathcal{B}}$.*

Theorem 3.6. *For each p, $1 \leqslant p \leqslant \infty$, $L^p[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ is a continuous dense embedding.*

Proof. By construction, $KS^2[\mathbb{R}^n]$ contains $L^1[\mathbb{R}^n]$ with continuous dense embedding, therefore we only need to show that $KS^2[\mathbb{R}^n] \supset L^p[\mathbb{R}^n]$ for $p \neq 1$. If $p < \infty$, $L^p[\mathbb{R}^n] \cap L^1[\mathbb{R}^n]$ is dense in $L^1[\mathbb{R}^n]$ and hence dense in $KS^2[\mathbb{R}^n]$. If $f \in L^p[\mathbb{R}^n]$ then:

$$\begin{aligned}\|f\|_{KS^2} &= \left[\sum_{k=1}^{\infty} t_k \left|\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right|^{\frac{2p}{p}}\right]^{1/2} \\ &\leqslant \left[\sum_{k=1}^{\infty} t_k \left(\int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^p d\mathbf{x}\right)^{\frac{2}{p}}\right]^{1/2} \\ &\leqslant \sup_k \left[\int_{\mathbf{R}^n} \mathcal{E}_k(\mathbf{x}) \left|f(\mathbf{x})\right|^p d\mathbf{x}\right]^{\frac{1}{p}} \leqslant \|f\|_p.\end{aligned}$$

Hence, $f \in KS^2[\mathbb{R}^n]$ and the embedding is continuous. For $p = \infty$, we have

$$\|f\|_{KS^2} = \left[\sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^2 \right]^{1/2}$$
$$\leqslant \left[\left[\sum_{k=1}^{\infty} t_k [vol(\mathbf{B}_k)]^2 \right] [ess\sup |f|]^2 \right]^{1/2} \leqslant M \|f\|_{\infty}.$$

Thus $f \in KS^2[\mathbb{R}^n]$, and $L^{\infty}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ as a continuous dense embedding. □

We can also construct $KS^p[\mathbb{R}^n]$ for all p by defining

$$\|f\|_{KS^p} = \begin{cases} \left\{ \sum_{k=1}^{\infty} t_k \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|^p \right\}^{1/p}, 1 \leqslant p < \infty, \\ \sup_{k \geqslant 1} \left| \int_{\mathbb{R}^n} \mathcal{E}_k(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right|, p = \infty. \end{cases}$$

It is clear that $\|\cdot\|_{KS^p}$ defines a new norm for $L^p[\mathbb{R}^n]$. If $KS^p[\mathbb{R}^n]$ is the completion of $L^p[\mathbb{R}^n]$ with respect to this norm, we have:

Theorem 3.7. *For each q, $1 \leqslant q \leqslant \infty$, $KS^p[\mathbb{R}^n] \supset L^q[\mathbb{R}^n]$ as a continuous dense embedding and, $KS^{\infty}[\mathbb{R}^n] \subset KS^p[\mathbb{R}^n]$ for each p.*

Remark 3.8. *We note that $KS^2[\mathbb{R}^n] = KS^2[\mathbb{R}^n]^* \supset L^1[\mathbb{R}^n]^{**} = \mathfrak{M}[\mathbb{R}^n]$, the space of finitely additive measures on $\mathbb{R}^n$, and in particular, the Dirac measure $\delta(\mathbf{x}) \in \mathfrak{M}[\mathbb{R}^n]$. The theory of distributions was developed via topological vector space methods because of the belief that there is no Hilbert space that contains the test functions $\mathcal{D}[\mathbb{R}^n]$. The next result shows that their mistake was due to a lack of interest in nonabsolutely integrable functions.*

Definition 3.9. *Let $C_c^{\infty}[\mathbb{R}^n]$ be the set of infinitely differentiable functions with compact support defined on $\mathbb{R}^n$. A sequence ϕ_j is said to converge to ϕ in the sequential compact support topology, if there exists a compact set $K \subset \mathbb{R}^n$ containing the support of $\phi_j - \phi$ for all j and, for every multi-index α, $D^{\alpha}\phi_j$ converges to $D^{\alpha}\phi$ uniformly on K. With this topology we call $C_c^{\infty}[\mathbb{R}^n]$ the space of test functions and denote it by $\mathcal{D}[\mathbb{R}^n]$.*

Theorem 3.10. *The test functions $\mathcal{D}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ as a continuous dense embedding.*

Proof. Because $KS^{\infty}(\mathbb{R}^n)$ is continuously embedded in $KS^2[\mathbb{R}^n]$, it is sufficient to prove the result for $KS^{\infty}(\mathbb{R}^n)$. Suppose that $\phi_j \to \phi$ in $\mathcal{D}[\mathbb{R}^n]$, so that there exists a compact set $K \subset \mathbb{R}^n$, containing the support of $\phi_j - \phi$ and $D^{\alpha}\phi_j$ converges to $D^{\alpha}\phi$ uniformly on K for every multi-index α. Let $L = \{l \in \mathbb{N} : \text{the support of } \mathcal{E}_l,\ stp\{\mathcal{E}_l\} \cap K \neq \emptyset\}$ and $M = \sup_{l \in L}[vol(\mathbf{B}_l)]$, then

$$\lim_{j\to\infty} \|D^{\alpha}\phi - D^{\alpha}\phi_j\|_{KS^{\infty}} = \lim_{j\to\infty} \sup_{l\in L} \left| \int_{\mathbb{R}^n} [D^{\alpha}\phi(x) - D^{\alpha}\phi_j(x)]\, \mathcal{E}_l(x)\, dx \right|$$
$$\leqslant M \lim_{j\to\infty} \sup_{x\in K} |D^{\alpha}\phi(x) - D^{\alpha}\phi_j(x)| = 0.$$

Thus, as the convergence is uniform, it follows that $\mathcal{D}[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$ as a continuous dense embedding. Because $KS^2[\mathbb{R}^n]$ is self-dual, we see that the Schwartz distributions, $\mathcal{D}^*[\mathbb{R}^n] \subset KS^2[\mathbb{R}^n]$. □

If $f, g \in L^1[\mathbb{R}^n]$, we denote the Fourier transform of f by $\mathfrak{F}(f)$ and the convolution of g with respect to f by $\mathfrak{C}_f(g)$.

Theorem 3.11. *The Fourier transform $\mathfrak{F}(\cdot)$ and, for each $f \in L^1[\mathbb{R}^n]$, the convolution with respect to f, $\mathfrak{C}_f(\cdot)$ both extend to $KS^2[\mathbb{R}^n]$ as bounded linear operators.*

Remark 3.12. *The possibility that $KS^2[\mathbb{R}^n]$ could exist was noticed by Gill and Zachary after reading a paper by Kuelbs [24]. It was constructed by V. Steadman and first appeared as a part of her dissertation (see [42]). As shown in the next section, this space is perfect for the Feynman path integral formulation of quantum mechanics.*

3.4 Feynman Path Integral

The properties of $KS^2[\mathbb{R}^n]$ derived earlier shows that it is a better replacement of $L^2[\mathbb{R}^n]$ for the study of the path integral formulation of quantum theory developed by Feynman. It is easy to prove that both the position and momentum operators have closed, densely defined extensions to $KS^2[\mathbb{R}^n]$. Furthermore, the extension of $\mathfrak{F}(\cdot)$ and $\mathfrak{C}_f(\cdot)$ by Theorem 3.11 ensures that both Schrödinger and Heisenberg theories have faithful representations on $KS^2[\mathbb{R}^n]$.

Because $KS^2[\mathbb{R}^n]$ contains $\mathfrak{M}[\mathbb{R}^n]$ the space of all finitely and countably additive measures as compact embeddings, it follows that all the approximating sequences for the Dirac measure converge strongly to it in the $KS^2[\mathbb{R}^n]$ topology. Thus, the finitely additive set function defined on the Borel sets $B \in \mathfrak{B}[\mathbb{R}^n]$ (Feynman kernel):

$$\mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ s, B] = \int_B (2\pi i(t-s))^{-1/2} \exp\{i|\mathbf{x}-\mathbf{y}|^2/2(t-s)\} d\mathbf{y}$$

is in $KS^2[\mathbb{R}^n]$ and $\|\mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ s, B]\|_{KS^2} \leqslant 1$ and

$$\mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ s, B] = \int_{\mathbb{R}^n} \mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ \tau, d\mathbf{z}]\mathbb{K}_{\mathbf{F}}[\tau, \mathbf{z};\ s, B], \quad \text{(HK-integral)}.$$

Definition 3.13. *Let* $\{\mathbf{P}_m = \{t_0^m, \tau_1^m, t_1^m, \tau_2^m, \cdots, \tau_m^m, t_m^m\}, m \leq [|\lambda t|] \ \}$ *be a family of HK-*δ_m *partitions of the interval* $[0,t]$, *where* $\mathbf{P}_k \subset \mathbf{P}_{k+1}$, *and* $\delta_{k+1}(t) \leq \delta_k(t)$, $\tau_0^m = 0$ *and for* $\psi \in KS^2[\mathbb{R}^n]$ *define: (we drop the superscript)*

$$\int_{\mathbb{R}^{n[0,t]}} \mathbb{K}_{\mathbf{F}}[\mathcal{D}_\lambda \mathbf{x}(\tau)\ ;\ \mathbf{x}(0)] = e^{-\lambda t} \sum_{k=0}^{[|\lambda t|]} \frac{(\lambda t)^k}{k!} \left\{ \prod_{j=1}^{k} \int_{\mathbb{R}^n} \mathbb{K}_{\mathbf{F}}[t_j, \mathbf{x}(\tau_j);\ t_{j-1}, d\mathbf{x}(\tau_{j-1})] \right\},$$

and

$$\begin{aligned} &\int_{\mathbb{R}^{[0,t]}} \mathbb{K}_{\mathbf{F}}[\mathcal{D}\mathbf{x}(\tau); \mathbf{x}(0)]\psi[\mathbf{x}(0)] \\ &= \lim_{\lambda \to \infty} \int_{\mathbb{R}^{n[0,t]}} \mathbb{K}_{\mathbf{F}}[\mathcal{D}_\lambda \mathbf{x}(\tau); \mathbf{x}(0)]\psi[\mathbf{x}(0)] \end{aligned} \tag{30}$$

whenever the limit exists.

The following is elementary, but a more general result covering almost all applications recently appeared in [8].

Theorem 3.14. *The function* $\psi(\mathbf{x}) \equiv 1 \in KS^2[\mathbb{R}^n]$ *and*

$$\int_{\mathbb{R}^{3[s,t]}} \mathbb{K}_{\mathbf{F}}[\mathcal{D}\mathbf{x}(\tau)\ ;\ \mathbf{x}(s)] = \mathbb{K}_{\mathbf{F}}[t, \mathbf{x};\ s, \mathbf{y}] = \frac{1}{\sqrt{2\pi i(t-s)}} \exp\{i|\mathbf{x}-\mathbf{y}|^2/2(t-s)\}.$$

Remark 3.15. *The above result is what Feynman was trying to obtain. However our approach is based on a photograph of the set of bubble tracks created*

by a particle as it traverses the chamber. Each τ_i is the center of a bubble and, for each t, λt is the average time between bubbles, while the Poisson distribution gives the probability that a given photograph has n bubbles.

Because $L^2[\mathbb{R}^n]$ is dense, $KS^2[\mathbb{R}^n]$ is separable and every orthonormal basis for $L^2[\mathbb{R}^n]$ may be used to form an orthonormal basis for $KS^2[\mathbb{R}^n]$. Thus they are unitarily equivalent; however.

1. *$KS^2[\mathbb{R}^n]$ contains both $\mathcal{D}[\mathbb{R}^n]$ and $\mathcal{D}^*[\mathbb{R}^n]$, whereas $L^2[\mathbb{R}^n]$ does not.*
2. *The integral of e^{ix^2} is finite in $KS^2[\mathbb{R}^n]$ but infinite in $L^2[\mathbb{R}^n]$.*

Thus, $L^2[\mathbb{R}^n]$ and $KS^2[\mathbb{R}^n]$ are mathematically equivalent but not physically equivalent.

4 Horwitz-Fanchi Historical Time

Newton suggested that the universe had an absolute definition of time that existed independently of any observer. He assumes that this clock provides a unique definition of simultaneity and ordering for events, but may only be understood as a mathematical object. (In Newton's era mathematics and personal philosophy still played a dominant role in physics). Einstein's special theory became problematic for Newton's absolute time, whereas Minkowski's postulate eliminated any sense of ordering. Feynman later suggested that we view a physical event as a 3-dimensional motion picture, exposing more of the outcome as the film unfolds (reintroduced the ordering property).

The first use of a fifth parameter to account for the missing order property was Fock, and later Stueckelberg, followed by Feynman and Schwinger. They each considered it a global clock in addition to the Minkowski space-time. Horwitz and Piron [21], and Fanchi and Collins gave it the name historical time. They were the first to realize that there should be physical justification for this clock (see [11]).

The purpose of this section is to prove this conjecture. Our proof uses the third definition of proper time and is based on a relatively weak assumption regarding the properties of the universe.

Theorem 4.1. *If the universe is representable in the sense that it is independent of any observer's particular portion of the universe, the universe has a unique clock.*

Proof. If Mc^2 is the total mass energy and H is the total energy Hamiltonian for the universe as seen by O, under the stated conditions H/Mc^2 is constant. By assumption this view is observer independent, so that O' will also obtain the same ratio. It follows from (3) that

$$d\tau = \frac{Mc^2}{H}dt = \frac{Mc^2}{H}dt', \quad \Rightarrow \int \tfrac{Mc^2}{H}dt = \int \tfrac{Mc^2}{H}dt' \equiv \tau_h$$

and τ_h uniquely defines a clock for the universe. □

The principle of homogeneity states that the distribution of matter does not vary significantly with its position on a large scale. Several recent studies have suggested that the principle of homogeneity is not as secure as believed in the past (see [30]). Recent studies have also led to questions about the principle of isotropy (see [39]). Thus, if future research supports these results, the cosmological principle may not hold true. However, the assumption of Theorem 4.1 can still remain valid.

4.1 The Feynman Interpretation of Antiparticles

One might naturally suspect that Dirac would have noticed that the positron could be interpreted as an electron moving backward in time after developing his hole theory. However, this idea was first developed by Feynman following the suggestion of Wheeler a few years before Stueckelberg introduced the same idea (for details, see Schweber [45]). However, the correct mathematical justifcation for such an interpretation only became possible in 1980's after the invention of dual numbers by Santilli (see [SN]). In this section, we show how use of these numbers makes Feynman's approach natural.

Formally, the real numbers is a quadruple $(\mathbb{R}, +, \cdot, \rightarrow)$, with 0 as the additive identity and 1 as the multiplicative identity. We have included $\rightarrow$ to include an unacknowledged sense of direction. If $\hat{\mathbb{R}} = \{-a : a \in \mathbb{R}\}$, then:

Definition 4.2. *The dual of the real numbers $(\hat{\mathbb{R}}, +, * \leftarrow)$ is a field, with $0 = \hat{0}$ as the additive identity, $-1 = \hat{1}$ as the multiplicative identity, and*

$$\hat{a} * \hat{b} = (-a)(-1)(-b) = (-ab) = \hat{ab} \text{ for all } \hat{a}, \hat{b} \in \hat{\mathbb{R}}.$$

Example 4.3. *The following example for quantum theory is as follows: the proper time evolution of a particle is defined on a Hilbert space $\mathcal{H}$ over complex numbers $\mathbb{C} = \mathbb{R} + i\mathbb{R}$, with Hamiltonian K and is given by the equation:*

$$i\hbar\frac{\partial\psi}{\partial\tau} = K\psi.$$

The conjugate equation is:

$$-i\hbar\frac{\partial\psi^*}{\partial\tau} = K\psi^*.$$

If we use $\hat{\mathbb{C}}$ as our number field, we can write the above equation as:

$$\hat{i} * \hat{\hbar} * \frac{\partial\hat{\psi}^*}{\partial\hat{\tau}} = \hat{K} * \hat{\psi}^*$$

This correctly defines an antiparticle as a particle with its proper time reversed, and with its evolution defined on $\mathcal{H}^*$ over $\hat{\mathbb{C}}$.

Remark 4.4. *It has been shown by Santilli that charge conjugation and isoduallity (as he calls it) are both equivalent for the particle-antiparticle symmetry operation.*

Discussion and Conclusion

In this study we have faced a number of foundational mathematical problems that have clouded our ability to correctly address some of the physical problems left over from the last century.

4.2 Classical Level

At the classical level, we have made Minkowski's unacknowledged third (mathematical) postulate for Einstein's special theory of relativity explicit, and provided direct proof that it is incompatible with the first postulate for two or more particles. We then used one of the three representations for the proper time to show how to resolve this incompatibility, allowing us to resolve the center of mass problem noticed by Pryce [37], the no Interaction problem first conjectured by Bakamjian and Thomas [2], and later proved by Currie, Jordan and Sudarshan [4]. This clarifies the way for a direct extension of Newton's theory to one compatible with that of Maxwell (addressed in the second paper).

4.3 Quantum Level

We first show that unconstrained by the Minkowski postulate, a new KG-equation is possible, which is valid for all spins. We then provide an analytical separation (diagonalization) of the Dirac equation into particle and

antiparticle components. The diagonalization is analytic in that it is achieved without transforming the wave functions and reveals the time nonlocal behavior of the particle-antiparticle relationship. The probability densities are also derived giving a complete decoupling of the spinor space $L^2[\mathbb{R}^3, \mathbb{C}^4] = L^2[\mathbb{R}^3, \mathbb{C}^2] \oplus L^2[\mathbb{R}^3, \mathbb{C}^2]$. We have then shown that the square root operator is nonlocal in space but related by a unitary transformation, implying that the mathematical definition of equivalence need not be physical.

4.3.1 Pauli Approximation

It is generally agreed that QED is in excellent agreement with experimental results. The fact that it is very successful is not in doubt. However, there was still some serious technical and foundational issues that need to be addressed. Historically, when Lamb and Retherford confirmed that the $2s_{1/2}$ state of hydrogen was shifted above the $2p_{1/2}$ state, and the Pauli approximation to the Dirac equation was used to determined that the Dirac equation was not sufficient. Thus, from a foundational perspective, it is important to understand the extent to which physical statements about the s-state behavior of the Dirac equation based on the Pauli equation (or any of its variants) are correct. We first recall that a Dirac particle has a finite probability of being at the origin in the s-state. This is not true for p-states or higher, so the Pauli approximation is justified. However, we explicitly show that the Pauli approximation is not valid for any calculation in the s-states, for two reasons.

1. We cannot replace $(E - V + mc^2)$ with $2mc^2$ because V is undefined at $r = 0$.

2. The correct probability density for the Pauli approximation is $\rho_\psi = |\psi(\mathbf{x})|^2 + \left|\frac{(\sigma \cdot \pi)}{2mc}\psi(\mathbf{x})\right|^2$ but it cannot be used to compute s-state perturbations because it contains an r^{-2} term making it undefined at $r = 0$.

We then show that these difficulties can be avoided, with an error of $\approx 10^{-5}$eV, provided a cut-off is imposed. An important conclusion is that a cut-off is required before QED. This implies that extra covariant formulations and renormalization procedures may not be necessary to obtain and justify the excellent QED results.

4.4 Feynman Path Integral

Historically, the mathematics community has responded in a number of ways to the introduction of new mathematical ideas or methods from the outside. The first response is to fit it into an existing framework. The second and most exciting response is when such an idea or method leads to the development of a new branch of mathematics. The most prevalent and successful response has been to find an existing mathematical structure that reasonably accommodates the physical theory and provides (at least) a framework for mathematical rigor. In some rare but important instances, no obvious mathematical structure can completely accommodate the theory in the manner presented by physicists. In this case, mathematicians have extended and/or adapted an existing mathematical theory, developed new mathematical structures or suggested (in frustration) that any conclusions derived from the use of these ideas or methods are suspect. Over the last 70 years, all of the above positions have appeared in response to to the introduction of Feynman's path integral into quantum theory.

In this study, we have introduced a more general integral (HK) and designed a new Hilbert space ($KS^2[\mathbb{R}^n]$) for the Feynman theory. This approach provides the correct framework for mathematical rigor without removing the physically intuitive and computationally effective approach suggested by Feynman. This space is also unitarily equivalent to $L^2[\mathbb{R}^n]$ showing that unitary equivalence is not always indicative of the same physics.

4.5 Feynman's Interpretation of Antiparticles.

In the last section, we introduced the dual of the real numbers and used them to provide mathematical meaning for Feynman's idea of anti particles with their proper time reversed.

Acknowledments

We would like to thank Professors L. P. Horwitz, E. Leib and R. M. Santilli for their continued interest and helpful comments.

Declaration

The authors certify that:

1. There is no data developed or attached to this paper.
2. They have no relevant financial or non-financial interests to disclose.
3. There are no conflicts of interest to declare that are relevant to the content of this manuscript.
4. The authors have no financial or proprietary interests in any material discussed in this manuscript.
5. They have no affiliations with or involvement in any organization or entity with any financial or non-financial interest in the subject matter or materials discussed in this manuscript.

References

[1] H.R Brown, Eur. J. Phys. **26**, S85 (2005).

[2] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953).

[3] K.M Case, Phys. Rev. **111**, (1958) 1011.

[4] D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963).

[5] P. A. M. Dirac, Proc. Roy. Soc (London) **A117** (1928) 610, **A118** (1928) 351.

[6] A. Einstein, Ann. d. Phys. **17**, 891 (1905).

[7] A. Einstein, Jahrbuch Radioaktivitat **V**, 422 (1907) (Berichtigungen).

[8] G. Esposito and T. L. Gill, *On the physical and mathematical foundations of quantum physics via functional Integrals*, (to appear) Int. J. Mod. Phys. (2025) 2530011.

[9] A. Einstein and J Laub, Über die elektromagnetischen Grundgleichugen für bewegte Köper, *Ann. d. Phys.* **26** (1908), 532-540.

[10] A. Einstein and J Laub, Über die im elektromagnetischen Felde auf ruhende Köper ausgeübten ponderomotorischen Kräfte, *Ann. d. Phys.*, **26** (1908) 541-550.

[11] J. R. Fanchi, *Confronting the ENIGMA of TIME*, World Scientific, Singapore, (2023).

[12] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[13] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

[14] J. Glimm and A. Jaffe, *Quantum Physics. A Functional Integral Point of View*, Springer, N. Y., (1987).

[15] I. S. Gradshteyn and I. M. Ryzhik Tables of Integrals, Series and Products, Academic Press, NewYork, (1980).

[16] T. L. Gill and W. W. Zachary, *Foundations for relativistic quantum theory I: Feynman's operator calculus and the Dyson conjectures*, Journal of Mathematical Physics **43** (2002), 69-93.

[17] T. L. Gill and W. W. Zachary, *Functional Analysis and the Feynman operator Calculus*, Springer New York, (2016).

[18] T.L. Gill, W.W. Zachary, *Analytic representation of the square-root operator*, J. Phys. A: Math. Gen. **38** (2005) 1-18.

[19] T.L. Gill, W.W. Zachary, and M. Alfred, *Analytic representation of the Dirac equation*, J. Phys. A: Math. Gen. **38** (2005) 1-22.

[20] D. Hestenes, *Space Time Algebra*, Gordon and Breach Pub. New York, (1966)

[21] L. P. Horwitz and C. Piron, Helv. Phys. Acta **46**, 316 (1981).

[22] R. Henstock, *The General Theory of Integration*, Clarendon Press, Oxford, (1991).

[23] L. P. Horwitz, *On the significance of a recent experiment demonstrating quantum interference in time*, (to appear in Phys. Rev. Letters, see arXiv:quant-ph/0507044).

[24] J. Kuelbs, *Gaussian measures on a Banach space,* Journal of Functional Analysis **5** (1970), 354–367.

[25] J. Kurzweil, *Nichtabsolut konvergente Integrale*, Teubner-Texte zür Mathematik, Band **26**, Teubner Verlagsgesellschaft, Leipzig, (1980).

[26] H. A. Lorentz, *Archives Nerlandaises des Sciences Exactes Naturelles*, **25**, (1892).

[27] H. A. Lorentz, *The Theory of Electrons* B. G. Teubner, Leipzig, 1906; (reprinted by Dover, New York, 1952).

[28] H. Leutwyler. *A no-interaction theorem in classical relativistic hamiltonian particle mechanics*, Nuovo Cim., **37**, 556 (1965).

[29] G. Longhi, L. Lusanna, and J. M. Pons, J. Math. Phys. 30, 1893 (1989).

[30] Nathan J. Secrest; Sebastian von Hausegger; Mohamed Rameez; Roya Mohayaee; Subir Sarkar; Jacques Colin. *A Test of the Cosmological Principle with Quasars*, The Astrophysical Journal Letters, **908** (2) 2021): L51

[31] A. Pais, *Subtle is the lord: the science and life of Albert Einstein*, Oxford University Press (1982).

[32] W. Perret and G. B. Jeffery (translators, with additional notes by A. Sommerfeld), *The Principle of Relativity* by H. A. Lorentz, A. Einstein, H. Minkowski and H. Weyl, Dover, New York, 1952.

[33] H. Poincaré, *Sur la dynamique de l' électron*, Rendiconti del Circolo matematico Rendiconti del Circolo di Palermo, **21**, 129-176 (1906).

[34] D. L. Pursey, Nucl. Phys. **53**, (1964) 174.

[35] W. Pauli, *Über den Zusammenhang des Abschlusses der Elektronengruppen im Atom mit der Komplexstruktur der Spektren.* Zeitschrift für Physik. **31** (1): 765-783 (1925).

[36] W. Pauli, Handbuch der Physik 2nd ed. **24**, (1933).

[37] M.H.L. Pryce, Proc. Roy. Soc. London A **195**, 400 (1948).

[38] S. G. Samko, *Hypersingular Integrals and Their Applications*, Analytical Methods and Special Functions, Vol. **5** Taylor & Frances, Pub., (2002).

[39] Saadeh D, Feeney SM, Pontzen A, Peiris HV, McEwen, JD . *How Isotropic is the Universe*, Physical Review Letters. **117** (13) (2016) 131302.

[40] L. I. Schiff, 1955 *Quantum Mechanics*, McGraw-Hill, New York (1955).

[41] J. C. Slater, *Quantum Theory of Atomic Structure*, Vol. II McGraw-Hill, New York (1960).

[SN] R. M. Santilli, *Isonumbers and genonumbers of dimension 1, 2, 4, 8, their isoduals, and pseudoduals, and hidden numbers of dimension 3, 5, 6, 7*, Algebras, Groups and Geometries, Vol. **10**, 273?322 (1993).

[42] V. Steadman, *Theory of operators on Banach spaces*, Ph.D thesis, Howard University, 1988.

[43] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That*, Benjamin, New York, (1964).

[44] S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Row, Peterson & Co., New York, (1961).

[45] S. Schweber, *QED And The Men Who Made It* , Princeton University Press, London (1994).

[46] S. Walters, In: Gonner, H., Renn, J., Ritter, J. (eds.) *The Expanding Worlds of General Relativity.* Einstein Studies, vol. 7, pp. 45-86. Birkhäuser, Boston (1999).

[47] A. O. Williams, Phys. Rev. **58** (1940), 723.

[48] S. Weinberg, *High energy behavior in quantum field theory,* Phys. Rev. **118** (1960), 838-849.